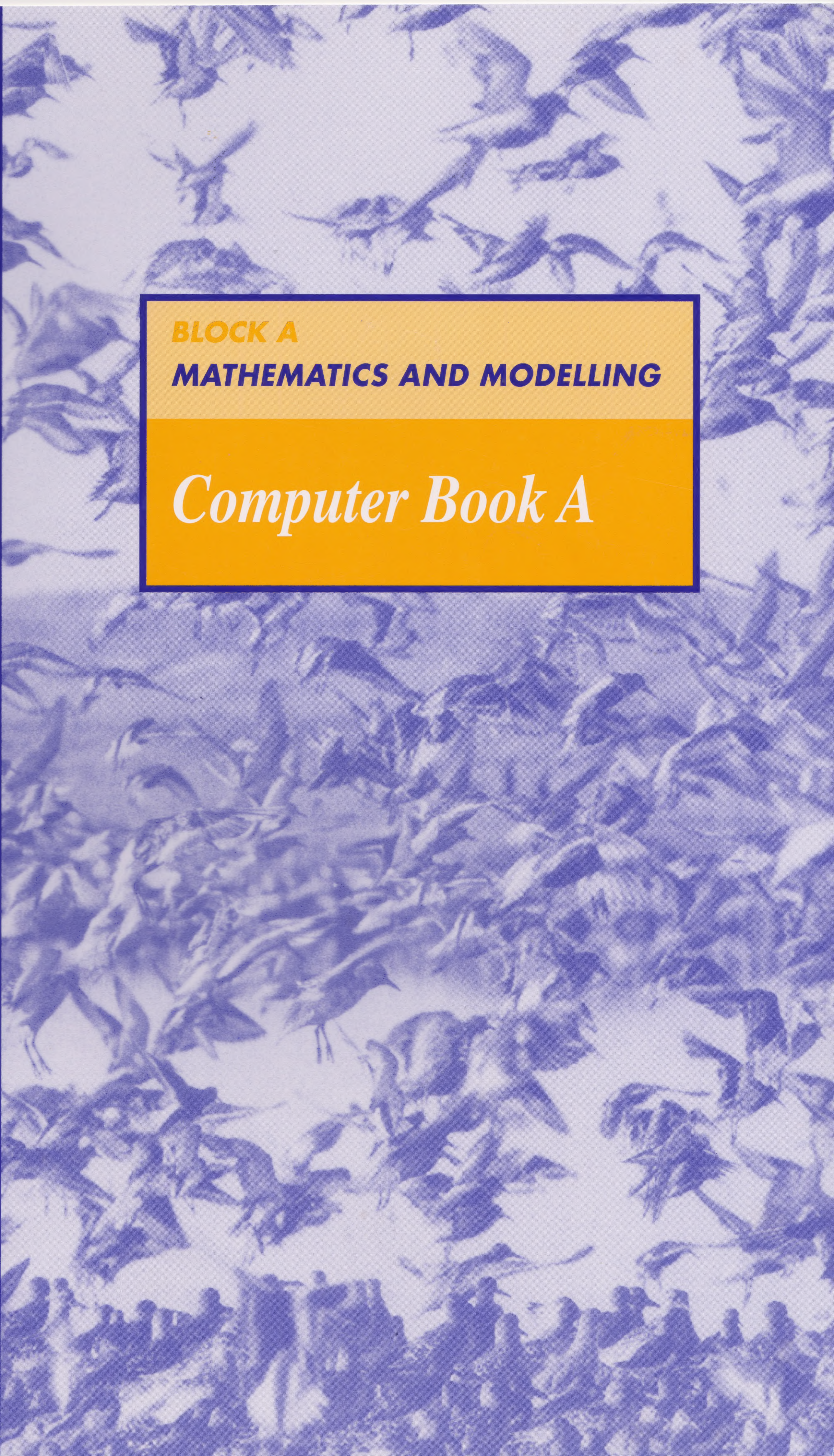
MST121 CB A

The Open University

A first level interdisciplinary course

Using Mathematics



BLOCK A

MATHEMATICS AND MODELLING

Computer Book A

COMPUTER BOOK

A

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BLOCK A
MATHEMATICS AND MODELLING

Computer Book A

Prepared by the course team

Using Mathematics

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A

About this course

This computer book forms part of the course MST121 *Using Mathematics.* This course and the courses MU120 *Open Mathematics* and MS221 *Exploring Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MST121 uses the software program Mathcad (MathSoft, Inc.) and other software to investigate mathematical and statistical concepts and as a tool in problem solving. This software is provided as part of the course.

This publication forms part of an Open University course. Details of this and other Open University courses can be obtained from the Student Registration and Enquiry Service, The Open University, PO Box 197, Milton Keynes, MK7 6BJ, United Kingdom: tel. +44 (0)870 333 4340, e-mail general-enquiries@open.ac.uk

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Contents

Guidance notes

This computer book contains those sections of the chapters in Block A which require you to use Mathcad. Each of these chapters contains instructions as to when you should first refer to particular material in this computer book, so you are advised not to work on the activities here until you have reached the appropriate points in the chapters.

In order to use this computer book, you will need the following Mathcad files.

Chapter A1

121A1-01 Sequences in Mathcad
121A1-02 Arithmetic sequences
121A1-03 Geometric sequences
121A1-04 Linear recurrence sequences
121A1-05 The mortgage sequence (Optional)

Chapter A2

121A2-01 Parametric equations of lines
121A2-02 Parametric equations of circles
121A2-03 Compound circular motion (Optional)

Chapter A3

121A3-01 Functions, graphs and solutions
121A3-02 The Mathcad solve block
121A3-03 Solving equations symbolically
121A3-04 Mathcad graph plotter

Instructions for installing these files onto your computer's hard disk, and for opening them, are given in Chapter A0.

Activities based on software vary both in nature and in length. Sometimes the instructions for an activity appear only in the computer book; in other cases, instructions are given in the computer book and on screen. Feedback on an activity is sometimes provided on screen and sometimes given in the computer book.

For advice on how each computer session fits into suggested study patterns, refer to the Study guides in the chapters.

Chapter A1, Section 6
Investigating sequences with the computer

In this section, you will use the computer to investigate the mathematical behaviour of arithmetic sequences, geometric sequences and linear recurrence sequences. The computer enables many terms to be computed quickly, graphs to be plotted easily and the long-term behaviour of sequences to be observed directly: by just changing a parameter, the terms of the sequence are recalculated and the graph redrawn. The long-term behaviour of linear recurrence sequences was discussed in Section 5, where the notation $a_n \to \infty$ as $n \to \infty$, and $a_n \to l$ as $n \to \infty$, was introduced.

There are five Mathcad files accompanying this section. The first file shows how to set up, display and plot graphs of sequences in Mathcad. The next three files cover the important mathematics of this section; they all have a similar structure and contain investigations of the behaviour of arithmetic, geometric and linear recurrence sequences, respectively. The final file, on the mortgage sequence, is entirely optional.

Assistance and explanations are provided on the pages within each file, with detailed Mathcad instructions where appropriate. In addition, help is available from the following.

◇ The MST121 reference manual, *A Guide to Mathcad.*
This contains full descriptions of all the Mathcad commands used in the files, including information about Mathcad error messages.

◇ Mathcad's own on-screen help facility.
Select the **Help** menu and **Mathcad Help**, then choose the 'Contents' tab for details of basic Mathcad operations, or the 'Index' or 'Search' tabs to search for a particular topic. (Please note that this information is not specific to MST121.)

6.1 Arithmetic sequences

The book sequence

Recall that the book sequence,

See Chapter A1, Section 2.

$$5, 8, 11, 14, \ldots, 38,$$

represents the total number of books that have been received by the end of successive months by a member of a book club. The closed form of this arithmetic sequence is

$$b_n = 5 + 3(n-1) \quad (n = 1, 2, 3, \ldots, 12).$$

In Activity 6.1, you are invited to set up this sequence on the computer, and plot a graph of it. This Mathcad file may look quite long, but all the tasks are straightforward.

Activity 6.1 Setting up sequences in Mathcad

Locate in the **Chapter A1** folder the file **121A1-01 Sequences in Mathcad**, and open it. Page 1 introduces the worksheet. Read pages 2 to 8, follow the instructions, and carry out Tasks 1 to 5.

Remember to make your own working copy of the file; see Chapter A0, Subsection 2.2, for guidance.

Comment

- ◇ Tasks 1 and 2 introduce subscripted variables and range variables, which play essential roles in setting up sequences in Mathcad.
- ◇ Task 3 involves defining a range variable and 12 subscripted variables to create the book sequence, and then using a table of values to display the sequence by entering $b =$. Such tables will be used throughout the course. Their appearance depends on the number of terms in the sequence – details of the different forms that tables can take are given in *A Guide to Mathcad.*

  Note that the Mathcad expressions can easily be modified if the rules of the book club change. For example, a change to sending four books each month, rather than three, could be handled by editing the definition for the subscripted variable $b_n := 5 + 3(n - 1)$ to read $b_n := 5 + 4(n - 1)$.
- ◇ Tasks 4 and 5 involve creating a graph and then formatting the graph display. You will meet these Mathcad techniques throughout the course – once again, the range variable plays an important part.

  Note that Mathcad calls the horizontal and vertical axes the x- and y-axes, although the graph is actually a plot of b_n on the vertical axis, against n on the horizontal axis. In fact, Mathcad refers to the whole graph as an 'X-Y Plot'. (This type of graph traditionally plots 'y against x', but any two variables can be plotted.)

Now close Mathcad file 121A1-01.

A general arithmetic sequence – varying the parameters

See Chapter A1, Section 2.

A general arithmetic sequence with recurrence system

$$x_0 = a, \quad x_{n+1} = x_n + d \quad (n = 0, 1, 2, \ldots),$$

where a is the first term and d is the common difference, has closed form

$$x_n = a + nd \quad (n = 0, 1, 2, \ldots).$$

Here, as with all the general sequences in this section, we start with the term x_0, rather than x_1, since this results in a simpler closed form. We obtain the *same terms* for the sequence, but with the subscripts decreased by one.

In particular, whether we start with x_0 or x_1 does not affect the long-term behaviour of the sequence.

Choosing the values of the parameters a and d determines a particular arithmetic sequence. In the next activity you will use Mathcad to investigate the effects of changing these parameters. Altering the parameters *one at a time* allows you to distinguish between the effects of changing each of them.

Activity 6.2 Investigating arithmetic sequences

Open Mathcad file **121A1-02 Arithmetic sequences**. Page 1 introduces the worksheet. Look at page 2, where the task starts with the arithmetic sequence obtained by setting $a = 10$ and $d = 1$, namely

$$x_n = 10 + n \quad (n = 0, 1, 2, \ldots, 20).$$

(a) Investigate the effect on the sequence of changing the parameter a, while keeping the parameter d constant (leave $d = 1$). Use the following values for a in turn:

$10, \quad 0, \quad -20.$

In an investigation of this type, it helps to choose round numbers and to change them in a systematic way, for example, from positive to negative values.

In each case, fill in the corresponding cell of the $d = 1$ row of the table in Figure 6.1 with a small sketch graph, and add a comment similar to the ones already filled in.

State briefly how altering a affects the graph of the sequence.

The sketch graphs here are intended to indicate the *overall* shape of the graph. We therefore draw line graphs, rather than plotting individual terms of the sequences.

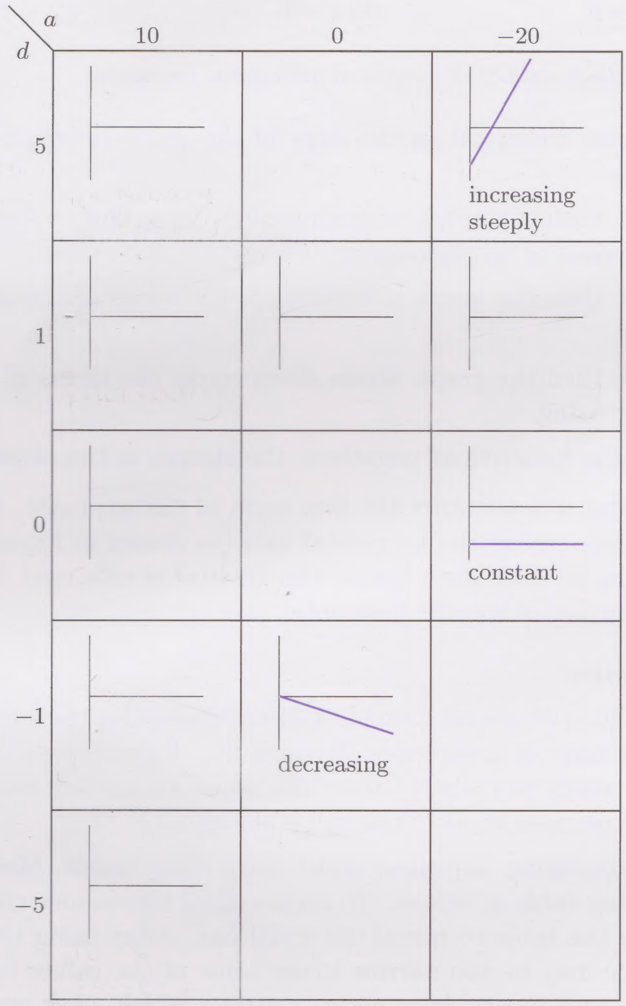


Figure 6.1 Sketch graphs of arithmetic sequences

(b) Investigate the effect on the sequence of changing the parameter d, while keeping the parameter a constant. Set $a = 10$. Then use the following values for d in turn:

$5, \quad 0, \quad -1, \quad -5.$

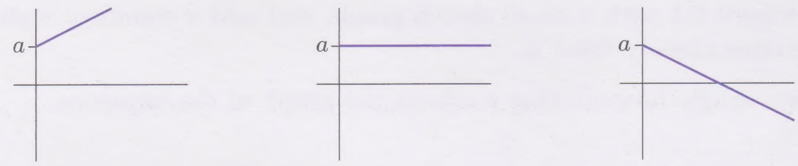
In each case, fill in the corresponding cell of the $a = 10$ column of the table in Figure 6.1.

State briefly how altering d affects the graph of the sequence.

Solutions are given on page 33.

Comment

The graph of an arithmetic sequence lies along a straight line. The graph can have one of the basic shapes in Figure 6.2. Each graph shown has $a > 0$.



(a) $d > 0$ (b) $d = 0$ (c) $d < 0$

Figure 6.2 Basic shapes of graphs of arithmetic sequences

The parameter d determines the slope of the graph (upwards or downwards).

These statements remain true if the first term is $x_1 = a$ and the common difference is d.

- ◇ If $d > 0$, then the graph slopes upwards (from left to right); the terms of the sequence are increasing.
- ◇ If $d = 0$, then the graph is horizontal; the terms of the sequence are constant.
- ◇ If $d < 0$, then the graph slopes downwards; the terms of the sequence are decreasing.

The larger d is (positive or negative), the steeper is the slope of the graph.

The parameter a determines the first term of the sequence. If $a > 0$, then the first term is above the horizontal axis (as shown in Figure 6.2 above); if $a = 0$, then the first term lies on the horizontal axis; and if $a < 0$, then the first term lies below the horizontal axis.

Mathcad notes provide extra information about the features and techniques used in the Mathcad files. They are *optional*.

Mathcad notes

- ◇ The built-in Mathcad variable *ORIGIN* specifies the subscript used for the first term of a sequence. By default, Mathcad sets *ORIGIN* to zero for every worksheet. Since the sequence x_n here has first term x_0, there is no need to alter the value of *ORIGIN*.
- ◇ When displaying sequences which have many terms, Mathcad provides a scrolling table of values. To access all of the values, click once on a value in the table to reveal the scroll bar. After using the scroll bar, the table may be too narrow to see some of the values fully. If you need to reset the table to an appropriate width, click once on a value in the table and choose **Calculate** from the **Math** menu (alternatively, press the `[F9]` function key).

Resizing a graph and fixing the graph scale are discussed in Chapter A2.

- ◇ The graph has been resized to make it appear larger on the page.
- ◇ The vertical (y-) axis scale has been fixed from -50 to 50.
- ◇ The graph trace has been formatted to display the individual terms of the sequence as blue crosses. On the vertical (y-) axis 'Auto Grid' has been switched off, and 'Number of Grids' has been set to 10, to improve the axis labelling. In addition, the 'Show Markers' option has been switched on for the vertical (y-) axis, with a dotted red marker line drawn at $y = 0$ to indicate the horizontal (x-) axis.

Now close Mathcad file 121A1-02.

6.2 Geometric sequences

A general geometric sequence with recurrence system

$$x_0 = a, \quad x_{n+1} = rx_n \quad (n = 0, 1, 2, \ldots),$$

where a is the first term and r is the common ratio, has closed form

$$x_n = ar^n \quad (n = 0, 1, 2 \ldots).$$

Choosing the values of the parameters a and r determines a particular geometric sequence. The long-term behaviour of the sequence x_n depends on the long-term behaviour of the sequence r^n, which is summarised in Table 6.1 for various ranges of r, including negative ones.

Remember that we are starting with the term x_0, because this gives a simpler closed form.

Notice that for $r = 0$ this closed form gives the correct value $x_0 = a$ only if we adopt the convention that $0^0 = 1$.

Table 6.1 Long-term behaviour of r^n

See Chapter A1, Section 5.

Range of r	Behaviour of r^n
$r > 1$	$r^n \to \infty$ as $n \to \infty$
$r = 1$	Remains constant: 1, 1, 1, ...
$0 < r < 1$	$r^n \to 0$ as $n \to \infty$
$r = 0$	Remains constant: 0, 0, 0, ...
$-1 < r < 0$	$r^n \to 0$ as $n \to \infty$, alternates in sign
$r = -1$	Alternates between -1 and $+1$
$r < -1$	Unbounded, alternates in sign

In the next activity, you will use Mathcad to investigate the effects of changing the parameters a and r of a geometric sequence. In particular, you will observe the long-term behaviour of r^n given in Table 6.1.

Activity 6.3 Investigating geometric sequences

Open Mathcad file **121A1-03 Geometric sequences**. Look at page 2 of the worksheet, where the task starts with the geometric sequence obtained by setting $a = 1$ and $r = 2$, namely

$$x_n = 2^n \quad (n = 0, 1, 2, \ldots, 20).$$

The graph on page 2 of the worksheet has a fixed vertical scale; only the terms of the sequence with values lying between -5 and 5 are shown. You can see from the table of values that the terms of this particular sequence quickly exceed 5. (Note that this table contains all of the values of the sequence – click on a value in the table to reveal the scroll bar.)

Now look at the graph on page 3 of the worksheet. This is a graph of the same sequence, but it rescales automatically so that the largest term (positive or negative) is always shown. This graph shows the long-term behaviour of the sequence, even if the later terms are very large (as is the case when $a = 1$ and $r = 2$), or very small. When interpreting sequence behaviour from this graph, it is important to look at the scale on the vertical axis.

To move quickly between pages of the worksheet, press `[Shift][Page Up]` and `[Shift][Page Down]`.

You may find it helpful to use both graphs in the activity, but it is best to base your sketch graphs on the one on page 2, which allows direct comparisons to be made between graphs of different sequences. In fact, you need to be on page 2 in order to alter the parameters a and r, so you should return there now.

Large numbers on the scale are displayed using scientific notation; for example, 500 000 is shown as $5 \cdot 10^5$. (A raised dot is used in place of $\times$ to indicate the multiplication.)

The sequence r^n grows very quickly when $r \geq 2$. We therefore choose values of r close to 1 in this study of the behaviour of geometric sequences.

(a) Investigate the effect on the sequence of changing the parameter a, while keeping the parameter r constant. Set $r = 1.1$. Then use the following values for a in turn:

$$2, \quad 1, \quad -1.$$

In each case, fill in the corresponding cell of the $r = 1.1$ row of the table in Figure 6.3.

State briefly how altering a affects the graph of the sequence.

Remember that these sketch graphs are intended to indicate the *overall* shape of the graph.

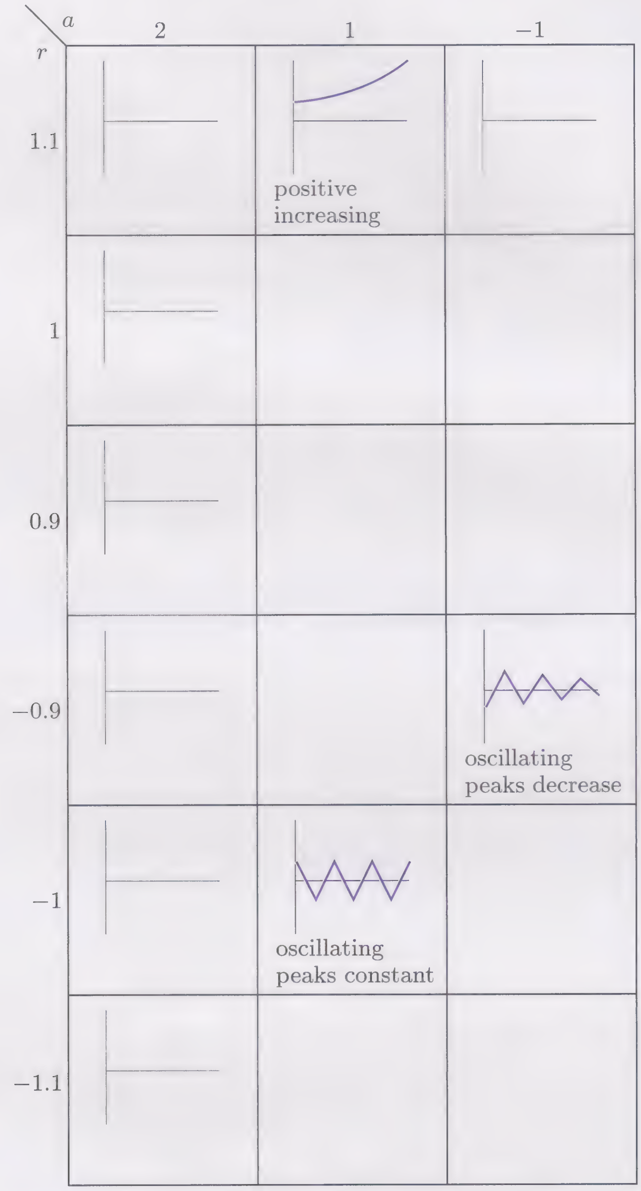


Figure 6.3 Sketch graphs of geometric sequences

(b) Investigate the effect on the sequence of changing the parameter r, while keeping the parameter a constant. Set $a = 2$. Then use the following values for r in turn:

$$1, \quad 0.9, \quad -0.9, \quad -1, \quad -1.1.$$

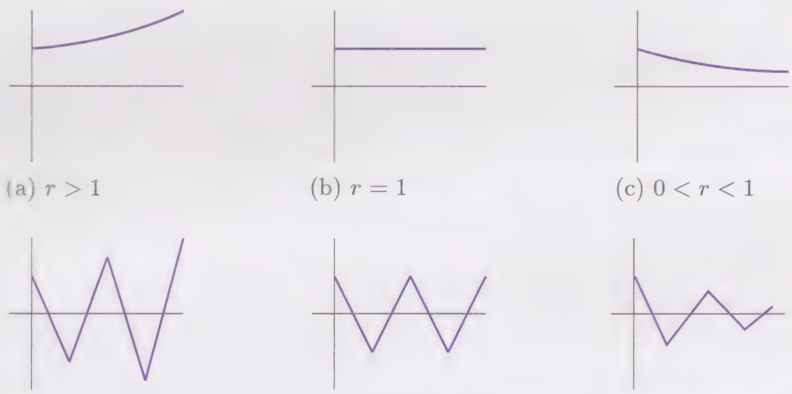
In each case, fill in the corresponding cell of the $a = 2$ column of the table in Figure 6.3. Try to predict the results before changing the parameter in the Mathcad worksheet.

State briefly how altering r affects the graph of the sequence.

Solutions are given on page 33.

Comment

The graph of a geometric sequence can have one of the basic shapes in Figure 6.4. Each graph shown has $a > 0$. The shapes for $a < 0$ are obtained by reflecting these graphs in the horizontal axis; that is, flipping them upside down.



(a) $r > 1$ (b) $r = 1$ (c) $0 < r < 1$

(d) $r < -1$ (e) $r = -1$ (f) $-1 < r < 0$

Figure 6.4 Basic shapes of graphs of geometric sequences

The parameter r determines the overall shape of the graph – whether it is constant, tends to infinity, flattens out, or oscillates with constant, increasing or decreasing peaks. The parameter a determines the first term of the sequence. If $a = 0$, then the graph is constant with all the terms equal to zero.

Mathcad notes

◇ If you change r to 0, then Mathcad gives the correct value for the first term, $x_0 = a$. When $r = 0$ and $n = 0$, Mathcad evaluates r^n to give 1.

In mathematics, the expression 0^0 is not generally defined, except where a convention is used. Mathcad, however, gives $x^0 = 1$ for *all* real x.

◇ The result format has been set to 'Number of decimal places' 6 and 'Exponential threshold' 9. These settings are used for all the results in the worksheet. However, Mathcad does not use these settings to display the numbers on the graph scales, where the exponential threshold is fixed (permanently) at 4.

◇ To make it easier to follow the order of the sequence, both graphs have been formatted to join the individual points (blue crosses) with a dotted line. To do this, the graph trace has been set to 'Symbol' ×'s, 'Line' dot, 'Color' blu, 'Type' lines and 'Weight' 1.

◇ The graph on page 3 always rescales automatically. Such automatic rescaling, in order to plot all the data, is Mathcad's default.

Now close Mathcad file 121A1-03.

6.3 Linear recurrence sequences

See Chapter A1, Section 4.

A general linear recurrence sequence is given by

$$x_0 = a, \quad x_{n+1} = rx_n + d \quad (n = 0, 1, 2, \ldots).$$

The overall shape of the graph of the sequence depends on the value of r. If $r \neq 1$, then the sequence has the closed form

$$x_n = \left(a + \frac{d}{r-1}\right) r^n - \frac{d}{r-1} \quad (n = 0, 1, 2, \ldots), \tag{6.1}$$

which is the sum of the geometric sequence

$$\left(a + \frac{d}{r-1}\right) r^n$$

and the constant sequence

$$-\frac{d}{r-1} = \frac{d}{1-r}.$$

The graph of such a linear recurrence sequence must therefore have one of the basic shapes shown in Figure 6.4, but displaced vertically by the amount $d/(1-r)$. On the other hand, a linear recurrence sequence with $r = 1$ is an arithmetic sequence with one of the basic shapes shown in Figure 6.2.

A particular linear recurrence sequence is obtained by assigning values to the parameters a, r and d. A systematic investigation of the effects of changing all *three* parameters would result in a great number of cases to consider, not to mention the problem of displaying the results in a three-dimensional table! We consider just one type of behaviour, where $0 < r < 1$ so $r^n \to 0$, and investigate the effect on this type of sequence of altering the other parameters, a and d.

Activity 6.4 Investigating linear recurrence sequences

Open Mathcad file **121A1-04 Linear recurrence sequences**. The layout of this worksheet is identical to that of the previous one, with the table and fixed-scale graph on page 2, and the graph that automatically rescales on page 3. There is one minor change in this worksheet, however. The parameter N allows you to change the number of terms of the sequence that are displayed. You can increase N if you feel that this would be helpful when determining the long-term behaviour, though it should be possible to see all the effects of changing a, r and d with the value of N provided.

Look at page 2, where the task starts with the linear recurrence sequence obtained by setting $a = 1$, $r = 0.5$ and $d = 1$, namely

$$x_n = -(0.5)^n + 2 \quad (n = 0, 1, 2, \ldots, 20).$$

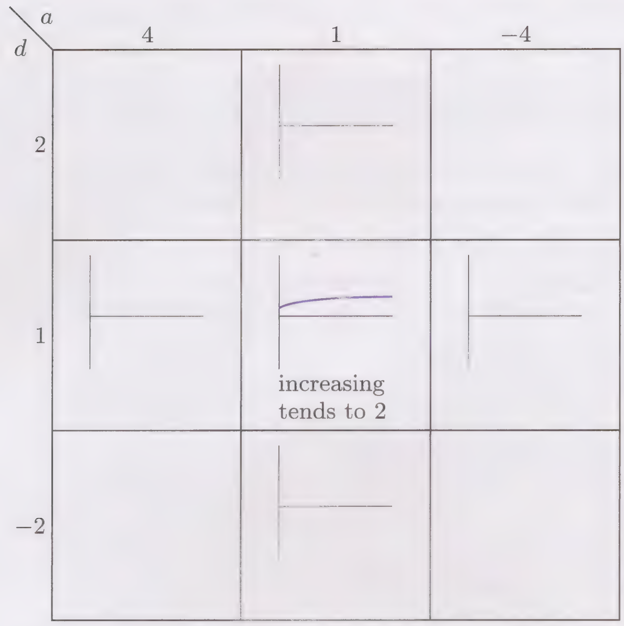
To change the parameters, you need to be on page 2.

(a) Investigate the effect on the sequence of changing the parameter a, while keeping the parameters r and d constant (leave $r = 0.5$ and $d = 1$). Use the following values for a in turn:

$$4, \quad -4.$$

In each case, fill in the corresponding cell of the $d = 1$ row of the table in Figure 6.5.

State briefly how altering a affects the graph of the sequence.



Remember that these sketch graphs are intended to indicate the *overall* shape of the graph.

Figure 6.5 Sketch graphs of linear recurrence sequences with $r = 0.5$

(b) Investigate the effect on the sequence of changing the parameter d, while keeping the parameters a and r constant (set $a = 1$ and leave $r = 0.5$). Use the following values for d in turn:

$2, \quad -2.$

In each case, fill in the corresponding cell of the $a = 1$ column of the table in Figure 6.5.

State briefly how altering d affects the graph of the sequence.

Solutions are given on page 34.

Comment

The parameter a has no effect on the long-term behaviour, but it does affect the initial term. The reverse is true for the parameter d. For $-1 < r < 1$, the graph of the sequence flattens out or the peaks get smaller and smaller, and the sequence tends to $d/(1 - r)$, which is the constant term in the closed form of equation (6.1).

Mathcad notes

◇ If you change the parameter r to 1, so $r - 1 = 0$, then Mathcad will give no terms for the sequence and will not plot a graph, except when the parameter d is 0. When $d = 0$, Mathcad evaluates the subexpression $d/(r - 1)$ in the closed form to give 0. When $d \neq 0$, Mathcad is unable to evaluate this, and gives an error.

In mathematics, the expression $0/0$ is not defined. Mathcad, however, gives $0/x = 0$ for *all* real x.

◇ Using a variable N to specify the number of terms in the sequence aids clarity and makes it easier to change things. Once N has been defined, it can be used as the final value in the definition for the range variable $n := 0, 1 .. N$ (on page 2) and to display the last term of the sequence as x_N (on page 3). Both of these can then be changed by simply changing the value in the definition for N. Note that, in Mathcad, N and n *are* two different variables. Note also that the range variable n runs from 0 to N, so we are actually calculating $N + 1$ terms of the sequence x_n.

Now close Mathcad file 121A1-04.

We conclude this section with the following optional activity.

Activity 6.5 Mortgage repayments (Optional)

See Chapter A1, Section 4.

You may like to use Mathcad file **121A1-05 The mortgage sequence** to experiment with repayments on a loan. The worksheet starts with the mortgage sequence, representing a loan of £10 000 to be repaid over a period of 20 years at an annual interest rate of 5%. You may like to try changing the size of the initial loan £L, the interest rate R%, the annual repayment £P, or the term of the loan N years.

Now close Mathcad file 121A1-05.

Chapter A2, Section 5 Parametric equations by computer

In this section, you will use the computer to plot lines and circles described by parametric equations.

There are three Mathcad files accompanying this section. The first and second files plot lines and circles, respectively. The final file is optional; it involves plotting more complicated curves that arise from compound circular motion.

5.1 Parametric equations of lines

Recall that a line $y = mx + c$, with slope m and y-intercept c, can be described by a pair of parametric equations

See Chapter A2, Subsection 4.1.

$$x = t, \quad y = mt + c.$$

This parametrisation can be used to plot the line in Mathcad, for particular values of m and c. The key requirement is to define a range variable to represent the parameter t. The graph is then constructed by entering the expressions describing x and y in terms of this range variable on the horizontal (x-) axis and vertical (y-) axis, respectively. This approach provides a powerful graphing technique in Mathcad, which can be used to plot *any* curve described by a pair of parametric equations.

Activity 5.1 Plotting a line described by parametric equations

Open Mathcad file **121A2-01 Parametric equations of lines**. Read page 2 of the worksheet, and carry out Task 1.

Solutions and some comments are included in the Mathcad worksheet.

Given two lines, it is possible to determine where they intersect using algebraic techniques. In the next activity, you are asked to find *approximately* where two lines intersect by identifying the intersection point on the graph using Mathcad's graph trace tool.

See Chapter A2, Section 1.

Activity 5.2 Finding intersections of two lines graphically

Read pages 3 and 4 of the worksheet, and carry out Tasks 2 and 3.

You should still be working with Mathcad file 121A2-01.

The solutions are provided in the worksheet as part of the comments following each task.

Comment

It is difficult to determine the *exact* point of intersection of two lines using this method. However, it is possible to find small ranges in which the coordinates of the point of intersection lie. For example, the lines in Task 3, with parametric equations $x = t$, $y = 2t + 3$ and $x = 4t - 7$, $y = -t + 16$, do intersect. This point of intersection has x-coordinate between 4.5 and 5.5, and y-coordinate between 12.5 and 13.5.

Note that the graph range has been chosen to include the point of intersection.

Remember that Mathcad notes are *optional*.

Note that a line is a type of curve, even though it is not 'curved'.

Mathcad notes

The techniques used to plot two curves on a graph (the two lines on page 3 of the worksheet) can be extended to plot three, four or more curves on the same graph. Simply enter all the expressions for 'x' separated by commas on the horizontal (x-) axis, and likewise the 'y' expressions on the vertical (y-) axis. Remember that Mathcad matches the expressions in pairs – the first 'y' expression is plotted against the first 'x' expression, the second against the second, and so on.

See Chapter A2, Example 4.2 and Activity 4.3.

In the main text, you saw that parametric equations for lines provide a straightforward method for determining the closest approach of two objects in linear motion. In the next *optional* activity, the computer is used to illustrate the tracing out of paths with respect to time, from which it can be seen whether two ships following intersecting courses actually collide.

Activity 5.3 Collision course? (Optional)

You should still be working with Mathcad file 121A2-01.

Look at page 5 of the worksheet, and follow the instructions.

Notice that the graph plots the two lines (the courses of the ships) and the right-hand endpoints of the lines (the positions of the ships), so there are four expressions on each axis. For a given value of T, the graph plots:

◇ the line $x = t$, $y = 2t + 3$ in red, and the line $x = 4t - 7$, $y = -t + 16$ in blue, both for the range $0 \leq t \leq T$;

◇ the point $(T, 2T + 3)$ as a red box symbol, and the point $(4T - 7, -T + 16)$ as a blue diamond symbol.

Mathcad notes

The square root symbol, used when calculating the distance between the two ships, can be obtained from the appropriate button on the 'Calculator' toolbar, or by typing \ (backslash). For example, you can type **\2=** to obtain $\sqrt{2} = 1.414$ (to 3 d.p.).

Now close Mathcad file 121A2-01.

5.2 Parametric equations of circles

See Chapter A2, Subsection 4.2.

Recall that a circle, with centre at (a, b) and radius r, has parametrisation

$$x = a + r\cos t, \quad y = b + r\sin t \quad (0 \leq t \leq 2\pi).$$

This parametrisation is ideal for plotting circles using Mathcad.

Activity 5.4 Plotting circles from parametric equations

Open Mathcad file **121A2-02 Parametric equations of circles**. Read page 2 of the worksheet, and carry out Task 1.

Solutions are given on page 3 of the worksheet.

Comment

You might have been surprised that none of the circles you obtained actually looks very circular! This is due to the way that Mathcad scales and sizes the graphs. All the curves do in fact represent circles, as you will see in Activity 5.5.

Mathcad notes

Both sine and cosine are built-in functions in Mathcad, and by default they work with angles in radians. They are obtained from the appropriate buttons on the 'Calculator' toolbar or by typing `sin` (or `cos`). The 'sin' and 'cos' buttons automatically insert a pair of round brackets after the function, that is, a left bracket, followed by an empty placeholder and a right bracket. Entry continues in this placeholder, and you can enter the value to which sin (or cos) is to be applied to obtain, for example, $\sin(t)$ or $\cos(\pi)$. If you wish to enter further information outside the brackets, then you will need to press `[Space]` to extend the editing lines around the brackets (and function) before you can do so. When entering the functions via the keyboard, you must type `sin` and `cos` using lower case letters, and you must also enter the brackets yourself, for example, type `sin(t)` or `cos([Ctrl][Shift]p)`.

When is a circle not a circle?

As you saw in Activity 5.4, a circle plotted by Mathcad may not always look like a circle. Consider the graphs below, all plotted over the range $t := 0, 0.01 \,..\, 2\pi$. Which of these are actually the graphs of circles?

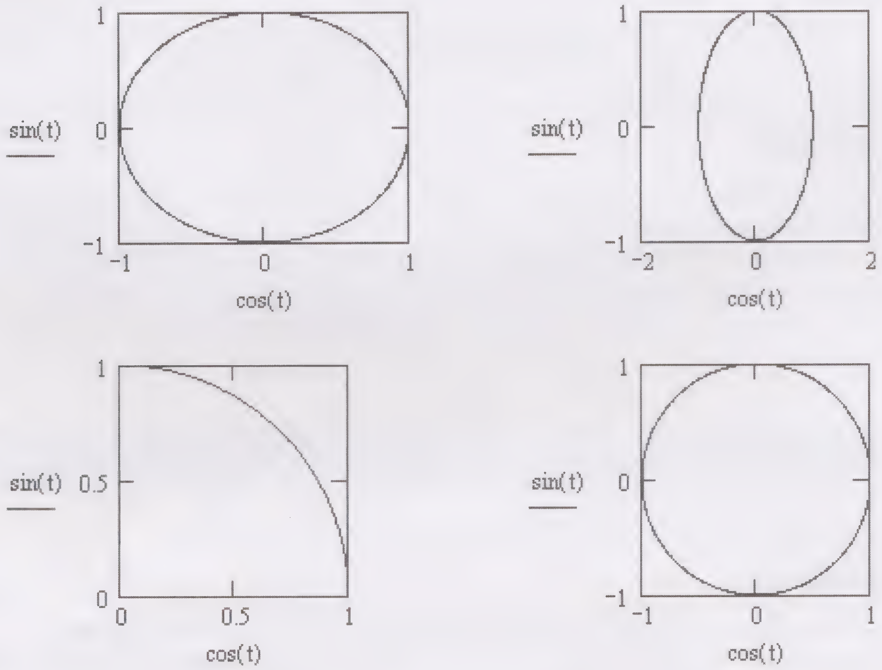


Figure 5.1 Circles plotted by Mathcad

Each of the graphs arises from these parametric equations of the unit circle:

$$x = \cos t, \quad y = \sin t \quad (0 \le t \le 2\pi).$$

The mathematical name for a stretched circle is an *ellipse*. Ellipses are studied in MS221.

The first graph is the default graph plotted by Mathcad – the curve is stretched horizontally to fit the rectangular graph box, and therefore appears to be an oval. The other three graphs have been resized to make their graph boxes square. They illustrate the effect of different scales on the appearance of a graph. The top right graph has the horizontal (x-) axis scale set from -2 to 2, while the vertical (y-) axis scale is set from -1 to 1. This results in the circle appearing to be squashed. The bottom left graph has both the horizontal (x-) axis and the vertical (y-) axis scales set from 0 to 1. This results in only one quarter of the circle being displayed. The final graph shows a circle which looks circular – the graph box is square and the scales of both axes are from -1 to 1, the default scales, thus showing the whole circle undistorted.

Note that Mathcad plots one point for each value of the range variable, and then joins them together with line segments. Therefore the step size chosen when defining the range variable plays a crucial role in the appearance of the curve plotted. For example, consider the following graph of the unit circle, where the range variable is set to $t := 0, 1 .. 2\pi$.

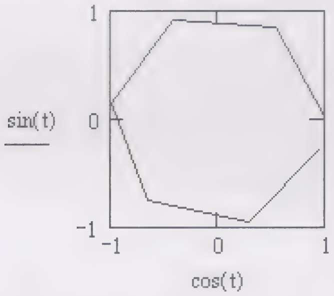


Figure 5.2 Mathcad plot of a circle where the step size is too large

As a rule of thumb, for a graph that is not a straight line, Mathcad should plot at least 100 points to be sure of obtaining a satisfactory picture; for example, the step size could be 0.1 for the range 0 to 10, and 0.05 for the range 0 to 2π.

These examples show that appearances may be deceptive. Mathcad always chooses a scale that allows it to display the whole curve, unless the scale is fixed manually. This may result in the graph being shorter and fatter, or taller and thinner, than the true picture. In the next activity you can practise resizing and fixing the scales of graphs to make them appear as you would expect.

Activity 5.5 How to resize a graph and fix the graph scale

You should still be working with Mathcad file 121A2-02.

Read pages 4 and 5 of the worksheet, and carry out Task 2.

Mathcad notes

◇ The instructions on how to fix the graph scale (on page 4 of the worksheet) relate to fixing the scale after a graph has been drawn. You can also fix the scale while you create a graph, when the graph box and all the placeholders are empty. (The four empty placeholders for the limits of the axes appear at the ends of the horizontal and vertical axes.)

◇ When formatting graphs, you may have noticed an 'Autoscale' option in the axis settings. This option applies only when Mathcad automatically sets the axis limits, and has no effect if you fix the graph scale by setting the axis limits yourself. When 'Autoscale' is on (the default), Mathcad sets the axis limits to round numbers. When it is off, Mathcad sets the axis limits to the extreme values of the data, so the traces plotted on the graph extend to the edges of the graph box.

The parametrisation representing motion in a circular path is

$$x = a + r\cos(kt), \quad y = b + r\sin(kt) \quad (T_1 \leq t \leq T_2),$$

See Chapter A2, Subsection 4.2.

where (a, b) is the centre of the circle and r is its radius. The non-zero constant k determines the rate and direction (anticlockwise or clockwise) of the motion, and the parameter range endpoints, T_1 and T_2, determine the portion of the circle traversed.

In the next activity you are asked to investigate the effects of altering the values of a, b and r on the position and size of the circle plotted, and the effects of altering the values of k, T_1 and T_2 on the portion of the circle plotted.

Activity 5.6 Exploring the parametrisation of a circle

Read page 6 of the worksheet, and carry out Task 3.

You should still be working with Mathcad file 121A2-02.

Mathcad notes

The range variable t is defined as $t := T1, T1 + 0.01 .. T2$. Thus t ranges from $T1$ to $T2$ in steps of size 0.01, where the values of $T1$ and $T2$ have already been defined. When defining range variables, it is important to remember that the three numbers in the definition are

starting value, next value .. final value.

We can avoid subscripts in Mathcad by using the variables $T1$ and $T2$ rather than T_1 and T_2.

The next activity involves parametric equations of both lines and circles. You will use the techniques that you have learned so far to find approximations to the points of intersection of a given line and circle.

Activity 5.7 Finding points of intersection graphically

Read page 7 of the worksheet, and carry out Task 4.

You should still be working with Mathcad file 121A2-02.

A solution is given on page 8 of the worksheet.

Comment

Notice that we needed to define *two* range variables here. The range variable t is used for the parametrisation of the circle, and the range variable u is used for the parametrisation of the line.

The letter most commonly used for a parametrisation is t. When other parameters are required, the letters s, u and v are often used.

Now close Mathcad file 121A2-02.

5.3 Two-circle compound motion (Optional)

Two-circle compound motion describes the position of a point determined simultaneously by *two* circles. Consider, for example, an electric food mixer with various attachments; see Figure 5.3. The point of attachment rotates about the central axis of the mixer (circle 1), and the attachment itself rotates about this point of attachment (circle 2).

The motion of a point on the edge of an attachment has the following type of parametrisation:

$$x = r\cos(kt) + R\cos(Kt), \quad y = r\sin(kt) + R\sin(Kt). \tag{5.1}$$

The values of r and k relate to circle 1, with r being the radius and k determining the rate of rotation. The values of R and K relate to circle 2 in a similar manner.

In a particular food mixer, an attachment rotates $3\frac{1}{3}$ times about the point of attachment every time this point rotates once about the central axis. So the ratio of K to k is 10:3. The radius r is a fixed distance, whereas R depends on the attachment.

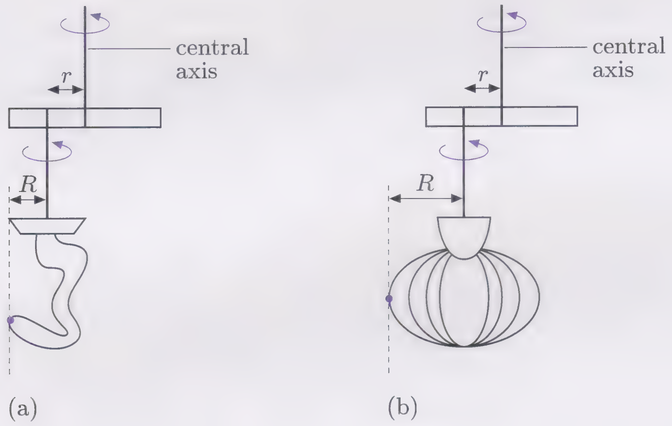


Figure 5.3 (a) Dough hook: $R = r$ (b) Whisk: $R = 2r$

On setting $r = 1$, $k = 3$, $R = 1$ and $K = 10$ in equation (5.1) for the dough hook, and $r = 1$, $k = 3$, $R = 2$ and $K = 10$ for the whisk, we see that the points plotted trace out the following curves.

Notice that the curve for the dough hook passes through the origin, but the curve for the whisk does not.

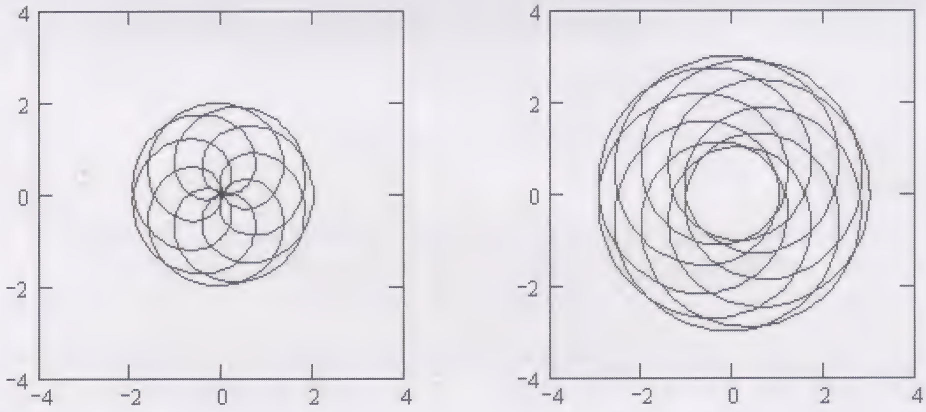


Figure 5.4 Curves traced by points on attachments: (a) dough hook, (b) whisk

You can create such graphs, associated with two-circle compound motion, in the final activity.

Activity 5.8 Compound circular motion (Optional)

Open Mathcad file **121A2-03 Compound circular motion**. Read the worksheet, and alter the values of r, k, R and K to investigate what patterns you can obtain.

Comment

If you wish to see how the dough hook and whisk patterns are built up by plotting just one revolution, then set

$$r = 1, \quad k = 1, \quad R = 1 \quad \text{and} \quad K = 10/3 \quad \text{for the dough hook,}$$
$$r = 1, \quad k = 1, \quad R = 2 \quad \text{and} \quad K = 10/3 \quad \text{for the whisk.}$$

Now close Mathcad file 121A2-03.

Chapter A3, Section 5
Functions, graphs and equations on the computer

In this section, you will use various Mathcad techniques to solve problems involving functions, graphs and equations.

See Chapter A3, Subsection 2.1.

There are four Mathcad files that accompany this section. In each of the first three files, a different technique is introduced, which is used first to solve the exhibition hall problem and then to solve a similar, but more complicated problem. The fourth file is set up as a general graph plotter. This can be used to plot graphs of the functions introduced in the main text, and also to solve two further problems concerning the *least* values taken by particular functions.

5.1 Problems and equations

First, here are three problems and their associated equations and functions.

The exhibition hall problem

Recall that the exhibition hall problem leads to the equation

$$4x^2 - 56x + 192 = 96. \tag{5.1}$$

All lengths used here are in metres.

Here x is the width of the border, the expression $4x^2 - 56x + 192$ represents the area of clear space in the exhibition hall, and 96 is half the total area. We represent the area of clear space using the function

$$f(x) = 4x^2 - 56x + 192 \quad (x \text{ in } [0, 6]), \tag{5.2}$$

so equation (5.1) can be written in the form $f(x) = 96$. Figure 5.1 shows the graph of $y = 4x^2 - 56x + 192$. The solid part is the graph of $y = f(x)$.

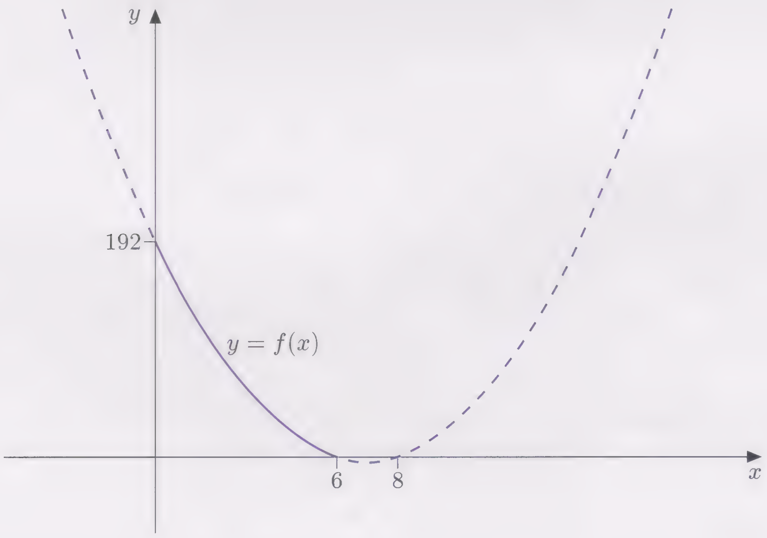


Figure 5.1 Graph of the function $f(x) = 4x^2 - 56x + 192$ (x in $[0, 6]$)

The modified exhibition hall problem

Suppose now that you have to find the width of the border in the exhibition hall that results in the clear space having area 144. This problem leads to the equation

$$4x^2 - 56x + 192 = 144; \tag{5.3}$$

that is, $f(x) = 144$, where f is the function given in equation (5.2).

The packing case problem

The following problem is a three-dimensional version of the exhibition hall problem, with area replaced by volume.

> **The Packing Case Problem**
> A packing case is lined with polystyrene in such a way that each of the six sides has lining of equal thickness, and the resulting cavity is one third of the volume of the box. The dimensions of the box are 3 metres by 1 metre by 1 metre. What is the thickness of the lining?

One of the six sides is the lid.

We apply the usual approach to solving this problem: we introduce variables, state their ranges, find a formula for the volume of the cavity, and then use this formula to write down an equation and a relevant function.

Figure 5.2 shows the box with the polystyrene lining and the resulting cavity. Here we have called the volume of the cavity V, and the thickness of the polystyrene lining x. Clearly, x must lie in the interval $[0, 0.5]$.

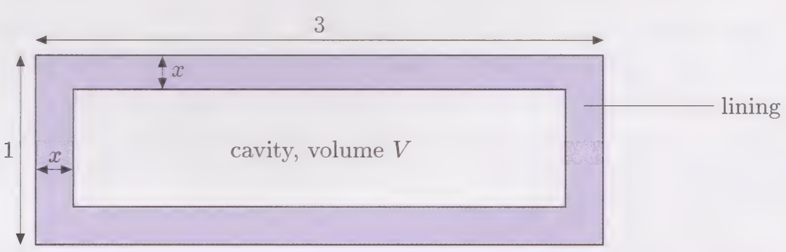


Figure 5.2 Variables for the packing case problem

Figure 5.2 represents a horizontal cross-section through the middle of the packing case.

The cavity has dimensions $3 - 2x$, $1 - 2x$ and $1 - 2x$, so its volume V is

$$\begin{aligned} V &= (3 - 2x)(1 - 2x)^2 \\ &= (3 - 2x)(1 - 4x + 4x^2) \\ &= 3 - 14x + 20x^2 - 8x^3. \end{aligned}$$

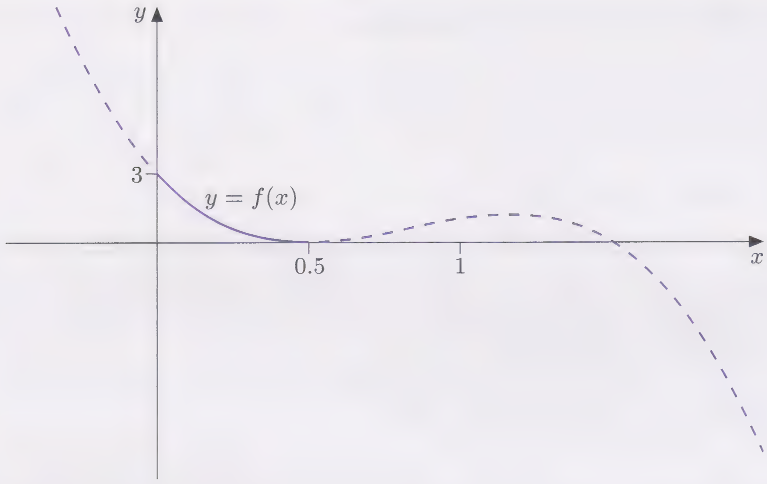
The problem is to find the value of x such that the volume V is one third of the volume of the box; that is, $V = \frac{1}{3} \times 3 \times 1^2 = 1$. Therefore we have to solve the cubic equation

$$3 - 14x + 20x^2 - 8x^3 = 1. \tag{5.4}$$

We represent the volume of the cavity using the cubic function

$$f(x) = 3 - 14x + 20x^2 - 8x^3 \quad (x \text{ in } [0, 0.5]). \tag{5.5}$$

Figure 5.3, overleaf, shows the graph of $y = 3 - 14x + 20x^2 - 8x^3$. The solid part is the graph of $y = f(x)$.



The graph of a cubic function may meet the x-axis once, twice (as here) or three times.

Figure 5.3 Graph of the function $f(x) = 3 - 14x + 20x^2 - 8x^3$ (x in $[0, 0.5]$)

5.2 Functions, graphs and solutions

In the first activity you are asked to define, in Mathcad, the function f given in equation (5.2), and then find various values of this function.

Activity 5.1 Finding values of a function

Open Mathcad file **121A3-01 Functions graphs and solutions**. Read pages 2 and 3 of the worksheet, and carry out Task 1.

Solutions and some comments are provided in the Mathcad worksheet.

In the next activity you will plot the graphs of $y = f(x)$, where f is the function in equation (5.2), and of the line $y = 96$, over the interval $[0, 6]$. The solution to the exhibition hall problem is the value of x at the point where these two graphs meet. Mathcad's graph zoom facility enables you to enlarge a portion of the plot in order to read off the coordinates of this point with reasonable accuracy.

Activity 5.2 Using a graph to solve an equation

You should still be working with Mathcad file 121A3-01.

Read pages 4 and 5 of the worksheet, and carry out Tasks 2 and 3. These show how to use Mathcad's graph zoom facility to find the solution to the exhibition hall problem.

Comments and a solution are provided in the Mathcad worksheet.

Mathcad notes

- ◇ The technique used to plot the line $y = 96$ can also be used to plot the x-axis, the line $y = 0$, on a Mathcad graph. The y-axis can be drawn in a similar manner, by plotting the line $x = 0$. Another method for adding axes is to format the graph and use 'Show Markers'. See *A Guide to Mathcad* for more details about both of these methods.
- ◇ When plotting two or more y-axis expressions against the same x-axis expression, you need to enter the x-axis expression only once. So the graphs in the worksheet could be obtained by typing just `x` in the x-axis placeholder and `f(x),96` in the y-axis placeholder.

The final page of the worksheet from file 121A3-01 is a graphical solution template. The word 'template' indicates that this part of the worksheet has been prepared so that you can use it to solve a variety of problems by making only a small number of changes. This reduces work, but it has the disadvantage that the notation used in the template may not be the same as that used in any given problem.

Activity 5.3 Using the graphical solution template

Read page 6 of the worksheet. This contains the graphical solution template, set up for the exhibition hall problem. Task 4 is to use this template to solve the modified exhibition hall problem and the packing case problem.

You should still be working with Mathcad file 121A3-01.

(a) Use the template to solve the modified exhibition hall problem. To do this, alter the target value to read

$$A := 144,$$

and then use Mathcad's graph zoom facility.

(b) Now use the template to solve the packing case problem. Set up the problem by altering the function to read

$$f(x) := 3 - 14x + 20x^2 - 8x^3,$$

the target value to read

$$A := 1,$$

and the interval endpoints to read

$$X1 := 0 \quad \text{and} \quad X2 := 0.5.$$

Then use Mathcad's graph zoom facility. (Note that you may need to click on 'Full View' in the 'X-Y Zoom' option box to obtain the new interval endpoints 0 and 0.5 in the graph.)

You can use the key sequence

```
3-14*x+20*x^2
[Space]-8*x^3
```

to create the right-hand side.

Solutions are given on page 34.

Comment

◇ Remember that in part (b) the target value A represents *volume*.

◇ In the definition of the range variable x, the step size used is 0.01. This is the horizontal displacement between the points plotted, so it is not worth zooming in to give an x-interval narrower than 0.01. If you wish to obtain a more accurate solution graphically, then you should first reduce the step size appropriately.

The step size also affects the accuracy of values obtained using the graph trace tool.

Now close Mathcad file 121A3-01.

5.3 Solve blocks

In Subsection 5.2 you solved the exhibition hall problem, the modified exhibition hall problem and the packing case problem, by solving an appropriate equation in each case, using the graph of a relevant function. The next activity introduces the Mathcad 'solve block', and uses it to solve these equations directly. Once again the worksheet includes a template, set up so that you can apply the solve block to a variety of equations.

Activity 5.4 Using the solve block template

Open Mathcad file **121A3-02 The Mathcad solve block**. Read page 2 of the worksheet, which explains how to create a solve block.

Now read page 3 of the worksheet. This contains the solve block template, set up to solve the exhibition hall problem. Check that the solution given is the one you would expect. Task 1 is to use this template to solve the modified exhibition hall problem and the packing case problem.

(a) Use the template to solve the modified exhibition hall problem. To do this, alter the target value to read

$$A := 144.$$

(b) Now use the template to solve the packing case problem. Set up the problem by altering the function to read

$$f(x) := 3 - 14x + 20x^2 - 8x^3,$$

the target value to read

$$A := 1,$$

and the interval endpoints to read

$$X1 := 0 \quad \text{and} \quad X2 := 0.5.$$

Then enter a 'guess' for the value of '$x :=$' in the interval $(0, 0.5)$.

Solutions are given on page 34.

Comment

- The Mathcad solve block uses an 'iterative' numerical method to solve the equation; that is, starting with the value entered for 'guess', it systematically refines guesses at a very fast rate to give an approximate solution, usually accurate to at least 2 decimal places. The initial guess may affect the solution obtained. Given the correct interval constraints, and a value for the guess within this interval, Mathcad will usually find an appropriate solution to the equation. If Mathcad cannot find a solution, then the expression '$Find(x) =$' appears in red, with the error message 'No solution was found ...'.
- Notice that the solve block uses *three* different forms of the 'equals' symbol, each with a different role.

  The symbol ':=' is used to *define* (assign a value to) the variable on its left. For example,

  $$x := 1$$

  assigns the value 1 to the variable x.

  The symbol '$\boldsymbol{=}$' is used to *equate* the expressions on either side of it. For example,

  $$4x^2 - 56x + 192 \boldsymbol{=} 96$$

  equates the expression $4x^2 - 56x + 192$ to the value 96.

  The usual symbol '=' is used to *evaluate* and *display* the expression to its left. For example,

  $$Find(x) = 0.917.$$

The bars of this equals sign are thicker than usual.

Now close Mathcad file 121A3-02.

5.4 Solving equations symbolically

The solve block is not the only way that Mathcad can solve equations. Some equations can be solved symbolically, that is, algebraically. This technique is used in the next two activities to solve the three problems in Subsection 5.1.

Activity 5.5 Solving quadratic equations

Open Mathcad file **121A3-03 Solving equations symbolically**.

(a) Read page 2 of the worksheet, and carry out Task 1, which uses the symbolic keyword 'solve' to solve the exhibition hall problem algebraically.

(b) Read page 3 of the worksheet, and carry out Task 2, which first uses the symbolic keyword 'solve' to solve the modified exhibition hall problem, and then shows how to obtain numerical rather than algebraic solutions.

Comments and solutions are provided in the worksheet, and further comments are below.

Comment

◇ Notice that the two solutions of the quadratic equation are displayed as a finite sequence in a column.

◇ There are no constraints on the solutions here, so the symbolic 'solve' gives every solution that it can find by using the quadratic equation formula. In each of parts (a) and (b), only one of the two 'solutions' found lies within the domain $[0, 6]$ of the function f given in equation (5.2).

◇ Some quadratic equations have no real solutions – that is, no solutions which are real numbers. In such cases, the symbolic 'solve' gives two solutions which are *complex numbers*. These involve i, a symbol with the property that $i^2 = -1$.

Complex numbers are studied in MS221.

In Activity 5.5, you obtained *exact* solutions, such as $7 + \sqrt{37}$ and $7 - \sqrt{37}$, to various quadratic equations. You also saw how to obtain the corresponding decimal solutions. The next activity asks you to apply a template for solving equations symbolically, which can be applied to quadratic or to more general equations.

Activity 5.6 Using the symbolic solution template

Look at page 4 of the worksheet, which contains a symbolic solution template. Use this template to solve the packing case problem, as follows.

You should still be working with Mathcad file 121A3-03.

(a) Set up the problem by altering the function to read

$$f(x) := 3 - 14x + 20x^2 - 8x^3$$

and the target value to be $A := 1$. The template then gives exact solutions to the equation $f(x) = 1$.

(b) Change the target value to $A := 1.0$ (that is, add a decimal point). Hence find, as a decimal, the solution to the packing case problem.

Comment

- The cubic equation $f(x) = 1$ here has the exact solutions 1, $\frac{3}{4} + \frac{1}{4}\sqrt{5}$, $\frac{3}{4} - \frac{1}{4}\sqrt{5}$. Mathcad's symbolic 'solve' can solve any cubic equation to give three exact solutions (two of which may be complex numbers). In general, however, the formulas for the solutions of a cubic equation are very long and can fill more than a page! You can avoid them appearing on this template in other cases, by making sure that at least one number in any cubic equation includes a decimal point, so that the solutions are displayed as decimals.
- The numerical solutions of the equation $f(x) = 1$ are 1, 1.3090... and 0.1909.... Only one of these lies in the domain $[0, 0.5]$ that was specified, in equation (5.5), for the packing case function. Hence the solution of that problem is 0.191 to 3 decimal places.

Now close Mathcad file 121A3-03.

You have now used three Mathcad techniques to solve various equations. As a general rule:

- a graphical solution can be used to solve *any* equation, but it is limited by the accuracy to which solutions can be found;
- the solve block can be used to solve *most* equations, including polynomial equations;
- the symbolic 'solve' can be used to solve many polynomial equations (and some others), but it will not always succeed.

For any given equation, a combination of these techniques may be used. For example, it is often useful to think about the approximate location of solutions graphically before finding their values numerically or symbolically.

5.5 A general graph plotter

The final activity uses a general graph plotter, which enables you to plot the graphs of a wide variety of functions. For example, you can use it to plot graphs of all the functions in the main text. You can also use it to solve two new problems: the orienteer's problem and the forester's problem. These are somewhat different to the problems solved earlier. In each case, a function f is given whose domain is an interval, and you are required to find the value of x in that interval for which $f(x)$ takes the *least* value.

The orienteer's problem

The sport of orienteering involves running or walking through the countryside, navigating between successive checkpoints. Orienteers are often faced with the following type of problem.

The Orienteer's Problem

An orienteer, who can run at 16 kph on a straight path and at 8 kph in a forest, is at point A in the forest, 1 km away from the nearest point O on the path, and wishes to reach a point C on the path, which is 2 km from O, in the shortest time. At which point B should the orienteer aim to join the path?

Here kph means 'kilometres per hour'.

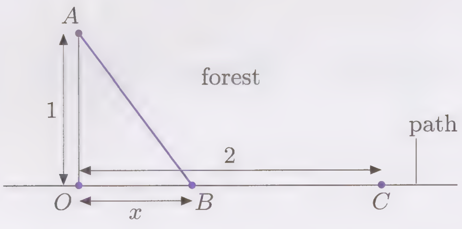


Figure 5.4 The orienteer's problem

All distances used here are in kilometres.

The variable x for the distance OB is introduced in Figure 5.4. The triangle AOB is a right-angled triangle, so AB is $\sqrt{1+x^2}$. Also, BC is $2-x$. Therefore the total time T taken (in hours) to run from A to C is

$$\begin{aligned} T &= \frac{\sqrt{1+x^2}}{8} + \frac{2-x}{16} \\ &= 0.125\sqrt{1+x^2} + 0.0625(2-x). \end{aligned}$$

We define the function f so that it represents the total time taken:

$$f(x) = 0.125\sqrt{1+x^2} + 0.0625(2-x) \quad (x \text{ in } [0,2]).$$

The orienteer's problem is to find the distance x that minimises the time taken to run from A to C. This involves finding the value of x in the interval $[0,2]$ that gives the least value of $f(x)$. The graph of $y = f(x)$ is shown in Figure 5.5, and it can be seen from this that the required value of x is slightly more than 0.5 km.

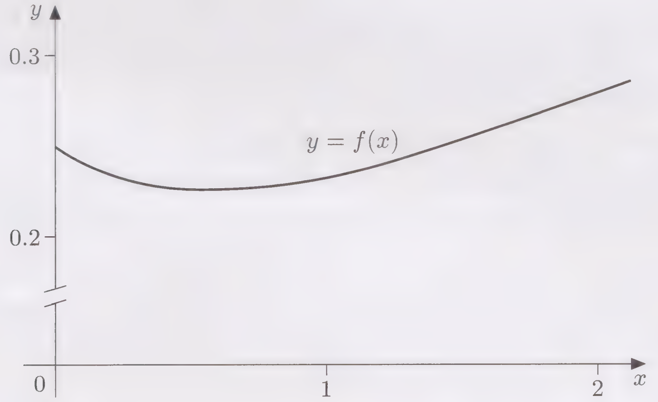


Figure 5.5 Graph of the function for the orienteer's problem

The forester's problem

A forester wishes to remove a row of felled trees in a forest, using a stationary tractor and a winch with a 100-metre cable; see Figure 5.6. Those trees nearest to the tractor are removed first. It is assumed that the trees are of similar size and are distributed evenly along the row.

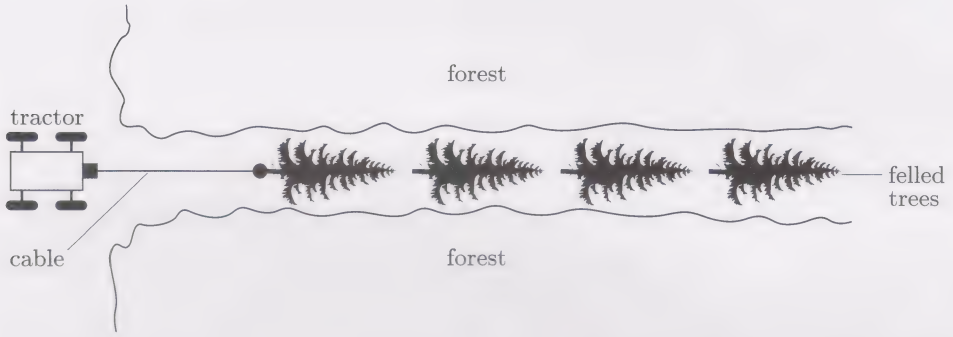


Figure 5.6 Removing a row of felled trees

> **The Forester's Problem**
> The cost (in pence) of removing a row, x metres long, of felled trees, where $0 \leq x \leq 100$, is modelled as
> $$600 + 6x + 0.9x^2. \qquad (5.6)$$
> For which value of x is the average cost per metre the least?

The cost formula in this model comprises three parts:

- ◇ the amount of 600 pence arises from fixed costs, independent of the length of the row x, such as the cost of preparing the tractor;
- ◇ the amount of $6x$ pence arises from costs which are the same for each tree removed, such as the time taken to attach the cable;
- ◇ the amount of $0.9x^2$ pence arises from costs which increase the further a tree is along the row, such as the time to carry the cable and the time to winch the tree to the tractor.

If it is known for which value of x the average cost per metre is least, then this knowledge might influence the possible locations of winching points in the forest. The above model is likely to be a great oversimplification, however.

The average cost per metre is

$$\frac{600 + 6x + 0.9x^2}{x} = \frac{600}{x} + 6 + 0.9x,$$

so we introduce the function

$$f(x) = \frac{600}{x} + 6 + 0.9x \quad (x \text{ in } (0, 100]).$$

The value 0 is excluded from the domain of f since the average cost per metre does not make sense for this value of x.

The forester's problem is to find the number x that minimises the average cost per metre. This involves finding the value of x in the interval $(0, 100]$ that gives the least value of $f(x)$. The graph of $y = f(x)$ is shown in Figure 5.7, and it can be seen from this that the required value of x is approximately 30.

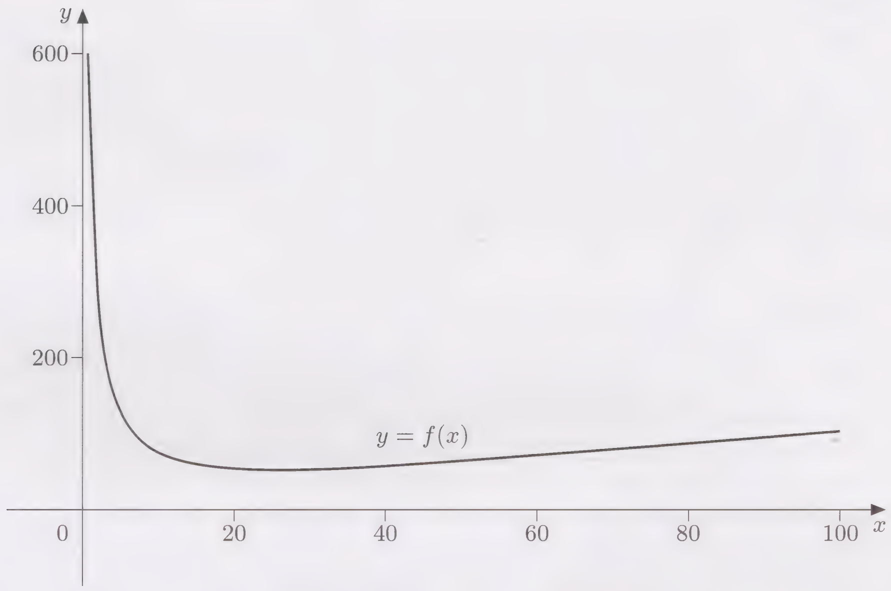


Figure 5.7 Graph of the function for the forester's problem

Activity 5.7 Using the Mathcad graph plotter

Open Mathcad file **121A3-04 Mathcad graph plotter**. The worksheet is set up to plot the function

$$f(x) = 0.125\sqrt{1+x^2} + 0.0625(2-x) \quad (0 \le x \le 2)$$

from the orienteer's problem. Task 1 is to use the graph plotter to solve the orienteer's problem and the forester's problem.

(a) Use the graph plotter to solve the orienteer's problem. To do this, use the graph zoom facility to zoom in on the portion of the plot where $f(x)$ takes the least value, and estimate the corresponding value of x by eye or with the graph trace tool.

(b) Now use the graph plotter to solve the forester's problem. Set up the problem by altering the function to read

$$f(x) := \frac{600}{x} + 6 + 0.9x,$$

and altering the interval endpoints to read

$$X1 := 1 \quad \text{and} \quad X2 := 100.$$

Then zoom in on the portion of the plot where $f(x)$ takes the least value, and estimate the corresponding value of x by eye or with the graph trace tool. (Note that you may need to click on 'Full View' in the 'X-Y Zoom' option box to obtain the interval endpoints 1 and 100 in the graph.)

You can use the key sequence

```
600/x[Space]+6+0.9*x
```

to create the right-hand side.

The point 0 is not in the domain of f, so we take $X1$ slightly to the right of this.

Solutions are given on page 34.

Comment

- ◇ After you alter the function in part (b), Mathcad gives a graph which appears to have a value close to 3×10^5 at about $x = 0$. In fact, Mathcad has omitted to plot the point $(0, f(0))$, because $f(0)$ is not defined. The first point plotted is $(0.002, f(0.002))$, where $f(0.002) \simeq 3 \times 10^5$ (see the second Mathcad note below). The graph looks like that in Figure 5.7 once $X1$ is set to the new value 1 and $X2$ to 100.
- ◇ Finding the least (or the largest) value of a function graphically may involve inspecting a part of the graph which is very flat. It can be tricky to judge accurately where the least value occurs – do your best!
- ◇ We encourage you to try using the graph plotter to plot graphs of the various functions introduced in the main text. The expressions for several of these functions can be input either by using buttons on the 'Calculator' toolbar or by typing in directly. Thus there are buttons for both $|x|$ and $\sqrt{x}$ on the toolbar, but these can also be typed in directly as `|x` (the vertical bar is obtained from `[Shift]\`) and as `\x`, respectively. The function 2^x can be obtained either by typing `2^x` or by using the 'x^y' toolbar button. Some standard functions cannot be obtained from toolbar buttons; for example, you need to type `acos(x)` in order to obtain $\arccos x$.

These alternatives were described for the functions sin and cos in the Mathcad notes for Activity 5.4 of Chapter A2, on page 17.

Mathcad notes

- ◇ The graph range is defined here as

$$x := X1, X1 + \frac{X2 - X1}{1000} \,..\, X2;$$

  that is, x ranges from $X1$ to $X2$ in steps of size $(X2 - X1)/1000$. This gives a way of plotting 1000 points, whatever values are chosen for the endpoints $X1$ and $X2$. As in Activity 5.3, this step size affects how far it is worth zooming in on the graph.

For example, in the forester's problem, the step size is

$$(100 - 1)/1000 = 0.099.$$

- ◇ If you try to plot the graph of a function f using a range variable x, and f is not defined at one of the values of x, then Mathcad avoids the problem by omitting any such value from the range.

Now close Mathcad file 121A3-04.

Chapter A1

Solution 6.2

For completeness, all the entries in the table have been included.

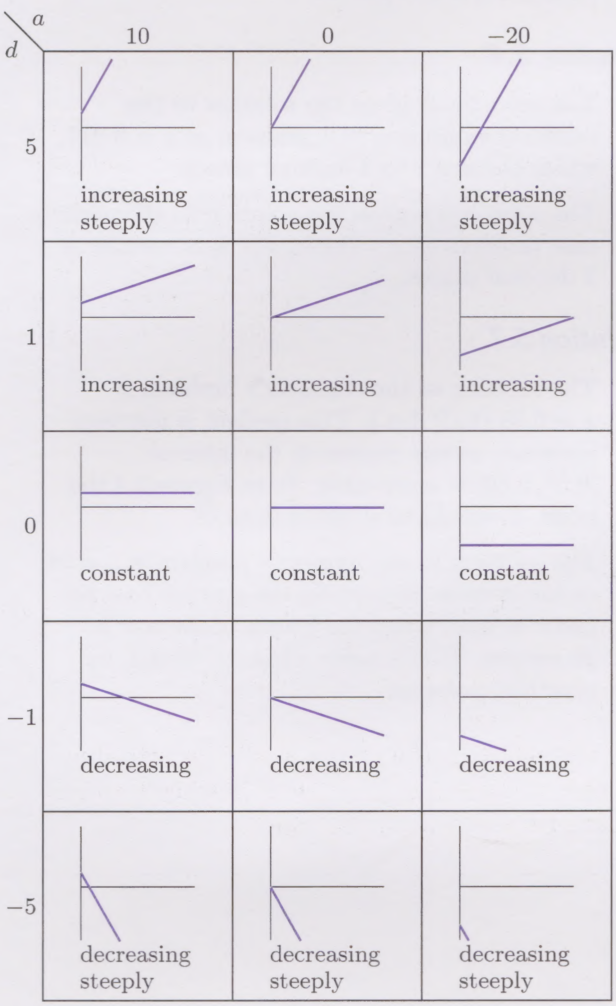


Figure S1.1 Sketch graphs of arithmetic sequences

(a) The parameter a is equal to the first term of the sequence; it affects where the graph starts.

(b) The parameter d determines the slope of the graph; that is, how steep the graph is (upwards or downwards).

Solution 6.3

The answers are included in the following table.

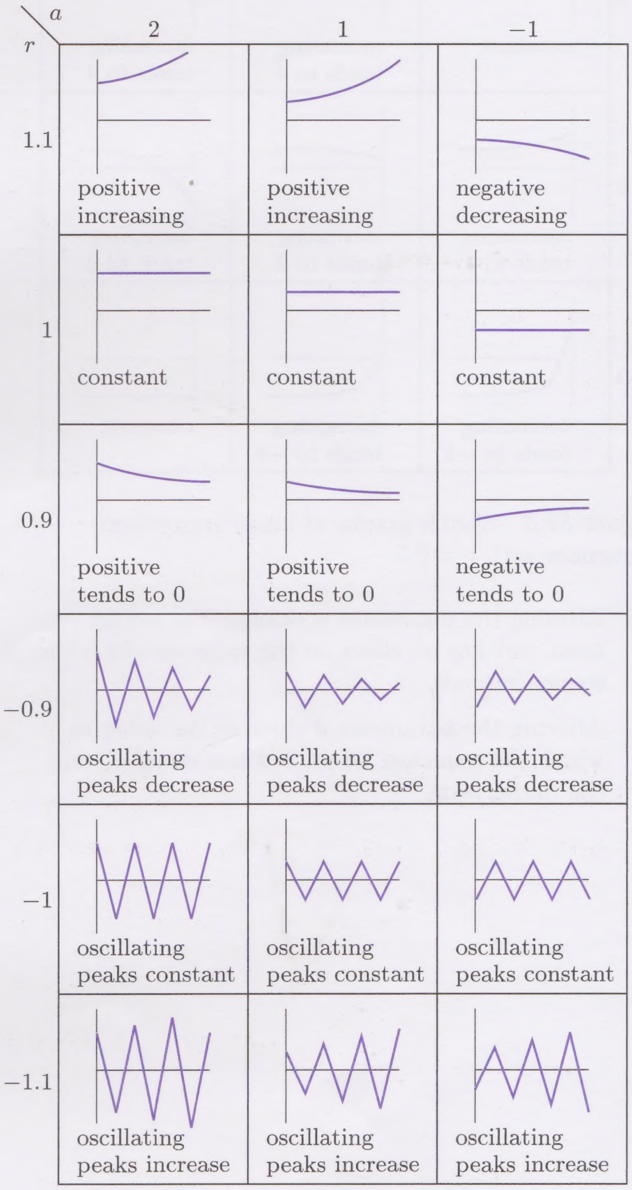


Figure S1.2 Sketch graphs of geometric sequences

(a) The parameter a is equal to the first term of the sequence; it affects where the graph starts and hence the orientation of the graph.

(b) The parameter r determines the *overall* shape of the graph – whether it is constant, tends to infinity, flattens out, or oscillates with constant, increasing or decreasing peaks.

Solution 6.4

The answers are included in the following table.

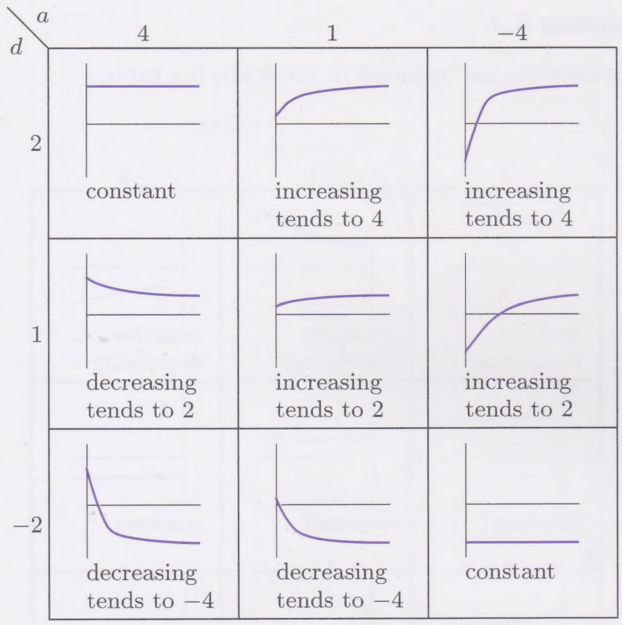


Figure S1.3 Sketch graphs of linear recurrence sequences with $r = 0.5$

(a) Altering the parameter a changes the initial term, but has no effect on the value to which the sequence tends.

(b) Altering the parameter d changes the value to which the sequence tends, but has no effect on the initial term.

Chapter A3

Solution 5.3

(a) The solution to the modified exhibition hall problem is $x = 0.92$ (to 2 d.p.). This method is not very accurate, so any answer in the interval $[0.91, 0.93]$ is acceptable.

(b) The solution to the packing case problem is $x = 0.19$ (to 2 d.p.). This method is not very accurate, so any answer in the interval $[0.18, 0.20]$ is acceptable.

Solution 5.4

(a) The solve block gives the solution to the modified exhibition hall problem as $x = 0.917$, which is correct to 3 decimal places.

(b) The solve block gives the solution to the packing case problem as $x = 0.191$, which is correct to 3 decimal places.

Solution 5.7

(a) The solution to the orienteer's problem is $x = 0.58$ (to 2 d.p.). This method is not very accurate, so any answer in the interval $[0.57, 0.59]$ is acceptable. So in Figure 5.4 the point B should be 0.58 km from O.

(b) The solution to the forester's problem is $x = 26$, to the nearest integer. So the average cost per metre is least when the length of the row is 26 metres. This is quite accurate enough for practical purposes.

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